EN5101 Digital Control Systems

Pulse Response

Digital Approximations

Stability

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Transfer Function of a Discrete System

transfer function in sampled systems.

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$$

using shifting theorem in z-transformation

$$Y(z) + a_1 z^{-1} Y(z) + \cdots a_n z^{-n} Y(z) = b_0 U(z) + b_1 U(z) z^{-1} + \cdots + b_m z^{-m} U(z)$$

$$(1 + a_1 z^{-1} + \cdots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}) U(z)$$

$$P(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Let's draw the controller structure

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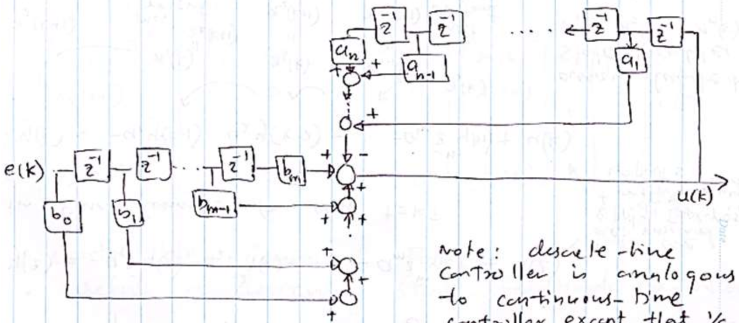
Structure of a Discrete Controller (Tra Fn)

$$e(k) \longrightarrow \boxed{C(z)} \longrightarrow u(k)$$

$$U(z) + a_1 z^{-1} U(z) + \cdots + a_n z^{-n} U(z) = b_0 E(z) + b_1 E(z) z^{-1} + \cdots + b_m z^{-m} E(z)$$



Note: discrete-time controller is analogous to continuous-time controller except that 1/s (integrators) are replaced by delay operators (z^{-1}).

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Tra Fn to State Space Model in DT

Example

$$\frac{Y(z)}{U(z)} = \frac{2}{1 + 2z^{-1} + 3z^{-2}}$$

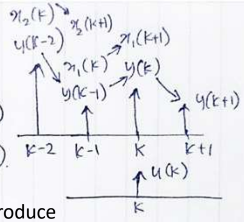
$$Y(z) = -2z^{-1} Y(z) - 3z^{-2} Y(z) + 2U(z)$$

z^{-1}

$$y(k) = -2y(k-1) - 3y(k-2) + 2u(k)$$

$x_1(k+1)$, $x_1(k)$, $x_2(k)$, $x_2(k+1)$

1. Introduce state variables



State Equations

$$x_1(k+1) = -2x_1(k) - 3x_2(k) + 2u(k)$$

$$x_2(k+1) = x_1(k)$$

2. Write state equations

SSM

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k).$$

3. Put it in matrix form

$$y(k) = x_1(k+1)$$

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Tra Fn to State Space Model in DT

$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_n z^{-n} Y(z) + U(z)$ ← determined by the Controller $U(z)$

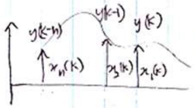
in time-domain for any $t = kT$.

$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_n y(k-n) + u(k)$

$\underbrace{x_1(k+1)}_{\text{output equation}}$ ← $\underbrace{x_1(k) = x_2(k+1) \quad x_2(k) = x_3(k+1) \quad \cdots \quad x_{n-1}(k) = x_n(k+1) \quad x_n(k)}_{\text{state equations}}$

$x_1(k)$: most recent value; $x_n(k)$: most distant value

assign discrete-time state variables $x_1(k), x_2(k) \ldots x_n(k)$



We can write state equations as follows

$x_1(k+1) = -a_1 x_1(k) - a_2 x_2(k) + \cdots - a_n x_n(k) + u(k)$

$x_2(k+1) = x_1(k)$

$x_3(k+1) = x_2(k)$

$\vdots$

$x_n(k+1) = x_{n-1}(k)$

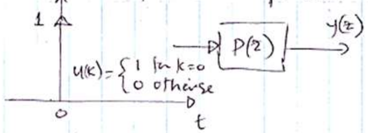
In matrix form,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

- Controllable Canonical Form.
- useful in designing state feedback Controller

Unit Pulse Response

unit pulse response (first order system)



$U(z) = \sum_{k=0}^{\infty} u(k) z^{-k} = u(0) z^{-0} = 1$

$Y(z) = P(z)\, U(z)$

$y(k) = Z^{-1}[P(z) . U(z)]$, $U(z) = 1$

$= Z^{-1}[P(z)]$ Pulse response of a system is the inverse z-transform of the system transfer function.

Eg: $P(z) = \frac{b_1 z}{z - a_1}$ first order (single pole)

pulse response

$y(k) = Z^{-1}[P(z)]$. Recall $Z[a^k] = \frac{z}{z-a}$

$= Z^{-1}\left[b_1 \frac{z}{z-a_1}\right]$

$= b_1 a_1^k$ exponential decay for $|a_1| < 1$

" rise for $|a_1| > 1$

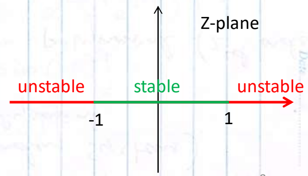
non oscillating if a_1 is +ve

oscillating if a_1 is −ve

size

sign

a_1	$\lvert\,\rvert > 1$	$\lvert\,\rvert < 1$
−ve	oscillating growing (unstable)	oscillating decaying (stable)
+ve	non oscillating growing (unstable)	non oscillating decaying (stable)



Unit Pulse Response – 2nd Order System

$$P(z) = z\frac{N(z)}{z^2+a_1 z+a_2}$$

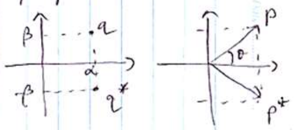
numerator polynomial

denominator polynomial (2nd order)

general 2nd order system has two poles, a pair of complex conjugate poles, thus can be resolved by partial fraction as follows

$$P(z) = z\left[\frac{q}{z-p} + \frac{q^*}{z-p^*}\right]$$ where q, q^* complex conjugate pairs, p, p^*

$$= z\left[\frac{\alpha+j\beta}{z-re^{j\theta}} + \frac{\alpha-j\beta}{z-re^{-j\theta}}\right]$$



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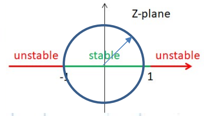
unit pulse response

$$u(z)=1 \longrightarrow \boxed{P(z)} \longrightarrow y(z)$$

System poles
$$z = re^{\pm j\theta} = r(\cos\theta \pm j\sin\theta)$$

$$y(k) = Z^{-1}\left[P(z)\right]$$
$$= Z^{-1}\left[(\alpha+j\beta)\frac{z}{z-re^{j\theta}} + (\alpha-j\beta)\frac{z}{z-re^{-j\theta}}\right]$$
$$= (\alpha+j\beta)Z^{-1}\left[\frac{z}{z-re^{j\theta}}\right] + (\alpha-j\beta)Z^{-1}\left[\frac{z}{z-re^{-j\theta}}\right]$$
$$= (\alpha+j\beta)(re^{j\theta})^k + (\alpha-j\beta)(re^{-j\theta})^k$$
$$= r^k\left[\alpha\left(e^{jk\theta}+e^{-jk\theta}\right) + j\beta\left(e^{jk\theta}-e^{-jk\theta}\right)\right]$$
$$= r^k\left[\alpha\, 2\cos(k\theta) + j\beta 2j\sin(k\theta)\right]$$
$$= \underbrace{2r^k}_{\text{magnitude}}\left[\underbrace{\alpha\cos(k\theta) - \beta\sin(k\theta)}_{\text{oscillation}}\right]$$

System is stable if $|r| < 1$ ⇒ poles stay within unit circle



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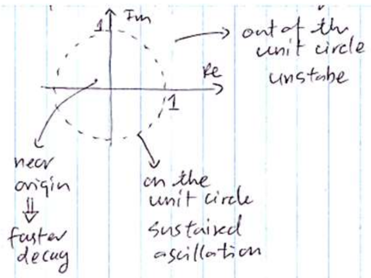
Pole Radius and Decay rate



Response magnitude = $2r^k$

Quiz: Determine the number of samples for the response to decay down to 1% of the initial magnitude

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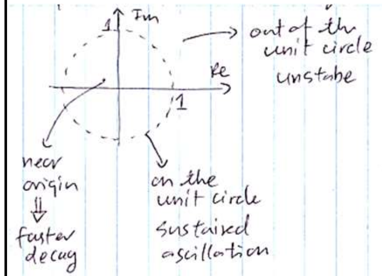
Pole Radius and Decay rate



Response magnitude = $2r^k$

samples to decay to 1% of the initial value

magnitude at $k=0$ = 2

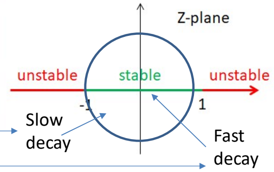
" " $k=n$ = $2r^n$

$$\frac{|\;|_{k=n}}{|\;|_{k=0}} = r^n$$

$$0.01 = r^n$$

$$\frac{\log 0.01}{\log r} = n$$

r	# Samples to 1%
0.9	43
0.8	21
0.6	9
0.4	5



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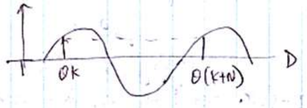
Pole Angle and Oscillation

\# samples per period $= N$ then

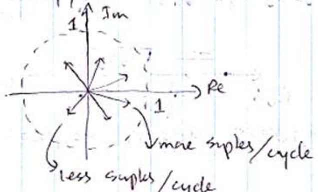
$\cos(\theta k) = \cos\{\theta(k+N)\} \Rightarrow \theta N = 2\pi$

$N = \frac{2\pi}{\theta}$



θ smaller $\rightarrow$ more samples per period.

θ bigger $\rightarrow$ less " " "



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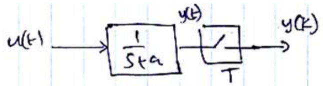
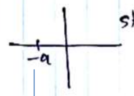
Effect of Sampling on Pole Locations



1st order System

$\frac{Y(s)}{U(s)} = \frac{1}{s+a}$ pole at $s=-a$

s - plane



stable if $a \geq 0$

$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+a}U(s)\right]$

unit impulse response $U(s) = 1$.

$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+a}\right]$

$= e^{-at}$

Continuous to discrete pole mapping

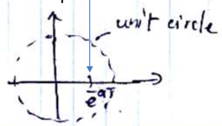
In descrete mode

$y(k) = y(t)\big|_{t=kT} \quad k=0,1,2\ldots$

$= e^{-aTk}$

$\mathcal{Z}$-tr $\quad Y(z) = \frac{z}{z-e^{-aT}}$ pole at $z = e^{-aT}$

z - plane



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Stable Or Unstable?

- CT system $\frac{1}{s+a}$ is stable $\forall\ a>0$. Then, the sampled system has to be stable as well
- As per the corresponding DT system $\frac{z}{z-e^{-aT}}$, stability criterion is

$$e^{-aT} \leq 1$$
$$\ln(e^{-aT}) \leq \ln 1$$
$$-aT \leq 0$$

which is true for any sampling interval T

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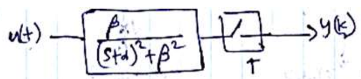
Second Order System

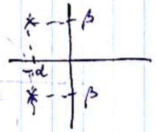


$\frac{Y(s)}{U(s)} = \frac{\beta}{(s+\alpha)^2+\beta^2}$ $\quad \Delta(s) = (s+\alpha)^2+\beta^2 = 0 \Rightarrow s_1 s_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+\beta^2)}}{2}$

$= -\alpha \pm j\beta$

unit impulse response $U(s) = 1$

$y(t) = \mathcal{L}^{-1}\left[\frac{\beta}{(s+\alpha)^2+\beta^2}\right] = e^{-\alpha t}\sin(\beta t)$ $\quad$ stable if $\alpha \geq 0$



In descrete mode

$y(k) = y(t)\big|_{t=kT} \quad k=0,1,2\ldots$

$= e^{-\alpha kT}\sin(\beta kT) \quad k \geq 0$

table $\mathcal{Z}$ $\quad Y(z) = \frac{z\,e^{-\alpha T}\sin(\beta T)}{z^2 - 2z\,e^{-\alpha T}\cos(\beta T) + e^{-2\alpha T}}$

$\Delta(z) = z^2 - 2z e^{-\alpha T}\cos(\beta T) + e^{-2\alpha T} = 0$

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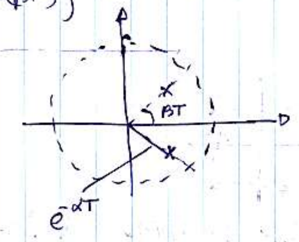
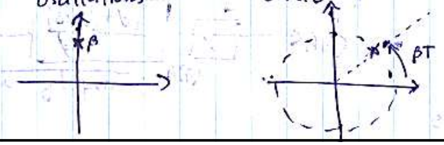
$$z_1, z_2 = 4e^{-\alpha T}\cos(\beta T) \pm \sqrt{4e^{-2\alpha T}\cos^2(\beta T) - 4e^{-2\alpha T}} \,/\, 2$$
$$= e^{-\alpha T}\left[\cos(\beta T) \pm j\sin(\beta T)\right]$$
$$= e^{-\alpha T}\, e^{\pm j\beta T}$$
$$= e^{(-\alpha \pm j\beta)T}$$



Case I $\alpha = 0$ radius $= 1 = e^{-\alpha T}$

Sustained oscillations

on the unit circle



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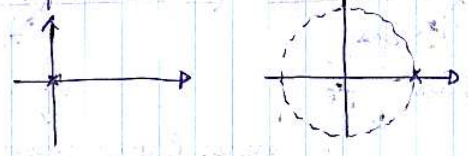
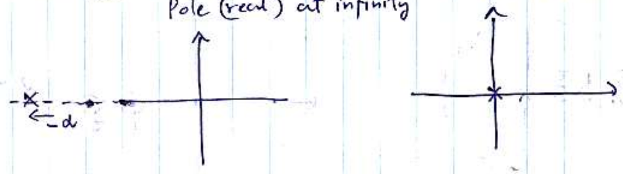
Case II $\alpha = 0$, $\beta = 0$ radius $= 1$ $e^{-\alpha T}$, angle $\beta T = 0$

rigid body dynamics



Case III $\alpha \to -\infty$ $\beta = 0$ Pole (real) at infinity

radius $= e^{-\alpha T} \Rightarrow 0$



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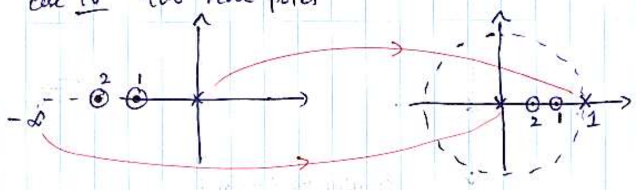
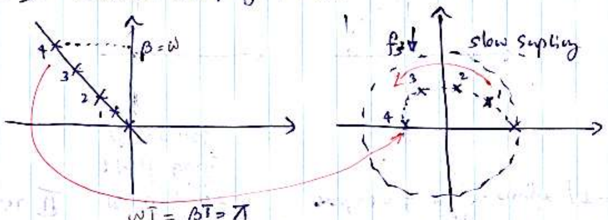
Case IV two real poles



Case V Constant damping ratio



$\omega T = \beta T = \pi$

one sample / half-period $\Rightarrow$ Nyquist limit

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Eg: Find the discrete time pole-locations of $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$

for $\zeta = 0.6$, $\omega_n = 5$ rad/s.

- check stability against Sampling interval T ($| \,| < 1$)
- check aliasing " " " T ($\angle < 180°$)

$$G(s) = \frac{5^2}{s^2 + 2\times0.6\times5\, s + 5^2} = \frac{25}{s^2 + 6s + 25}$$

$\Delta(s) = s^2 + 6s + 25 = 0 \Rightarrow$ poles $s_1 s_2 = \frac{-6 \pm \sqrt{36 - 4\times25}}{2} = -3 \pm j4$ (ω_d)

discrete-time poles $z_1 z_2 = e^{(s_1 s_2)T} = e^{(-3 \pm j4)T} = e^{-3T} \cdot e^{\pm j4T}$

$|\ |_{mag}$ $\angle_{ang}$

Alternatively

$z_1 z_2 = e^{-3T}(\cos 4T \pm j\sin 4T)$

$\omega T_P = 2\pi$

$T_P = \frac{2\pi}{\omega} = 1.5$

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Stability

$|e^{-3T}| < 1$ This is always true for any $T>0$

note: A stable $G(s)$ has stable $Z(s)$ under any T. however, slow sampling may cause aliasing in the feedback loop which will deteriorate the response.

for sampling at 4 Hz $T = 0.25$ s

$z_1 z_2 = 0.255 \pm j0.398$